A Simple Proof of Jung' Theorem on Polynomial Automorphisms of $\mathbf{C}^2$

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Abstract. The Automorphism Theorem, discovered first by Jung in 1942, asserts that if k is a field, then every polynomial automorphism of k^2 is a finite product of linear automorphisms and automorphisms of the form $(x,y) \mapsto (x+p(y),y)$ for $p \in k[y]$. We present here a simple proof for the case $k = \mathbf{C}$ by using Newton-Puiseux expansions.

1. In this note we present a simple proof of the following theorem on the structure of the group $GA(\mathbf{C}^2)$ of polynomial automorphisms of $\mathbf{C}^2$

Automorphism Theorem. *Every polynomial automorphism of* $\mathbf{C}^2$ *is tame, i.e. it is a finite product of linear automorphisms and automorphisms of the form* $(x,y) \mapsto (x+p(y),y)$ *for one-variable polynomials* $p \in \mathbf{C}[y]$.

This theorem was first discovered by Jung [J] in 1942. In 1953, Van der Kulk [Ku] extended it to a field of arbitrary characteristic. In an attempt to understand the structure of $GA(\mathbf{C}^n)$ for large n, several proofs of Jung's Theorem have presented by Gurwith [G], Shafarevich [Sh], Rentchler [R], Nagata [N], Abhyankar and Moh [AM], Dicks [D], Chadzy'nski and Krasi'nski [CK] and McKay and Wang [MW] in different approaches. They are related to the mysterious Jacobian conjecture, which asserts that a polynomial map of $\mathbf{C}^n$ with non-zero constant Jacobian is an automorphism. This conjecture dated back to 1939 [K], but it is still open even for $n = 2$. We refer to [BCW] and [E] for nice surveys on this conjecture.

2. The following essential observation due to van der Kulk [Ku] is the crucial step in some proofs of Jung' theorem.

Division Lemma: $F = (P,Q) \in GA(\mathbf{C}^2) \Rightarrow \deg P | \deg Q$ *or* $\deg Q | \deg P$.

Abhyankar and Moh in [AM] deduced it as a consequence of the theorem on the embedding of a line to the complex plane. McKay and Wang [MW] proved it by

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using formal Laurent series and the inversion formula. Chadzy'nski and Krasi'nski in [CK] obtained the Division Lemma from a formula of geometric degree of polynomial maps (f, g) that the curves $f = 0$ and $g = 0$ have only one branch at infinity. Here, we will prove this lemma by examining the intersection of irreducible branches at infinity of the curves $P = 0$ and $Q = 0$ in term of Newton-Puiseux expansions.

Our proof presented here is quite elementary and simpler than any proof mentioned above. It uses the following two elementary facts on Newton-Puiseux expansions (see, for example, [BK]).

Let $h(x, y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)$ be a reducible polynomial. Looking in the compactification $\mathbf{C}P^2$ of $\mathbf{C}^2$, the curve $h = 0$ has some irreducible branches located at some points in the line at infinity, which are called the *irreducible branchs at infinity*. For such a branch γ, the Newton' algorithm allows us to find a meromorphic parameterization of γ, an one-to-one meromorphic map $t \longmapsto (t^m, u(t)) \in \gamma$ defined for t large enough,

$$u(t) = t^m \sum_{k=0}^{\infty} b_k a t^{-k}, \ \gcd\{k : b_k \neq 0\} = 1,$$

The fractional power series $u(x^{\frac{1}{m}})$ is called a *Newton-Puiseux expansion at infinity* of γ and the natural number $\operatorname{mult}(u) := m$ - the *multiplicity* of u.

The first fact is a simple case of Newton's theorem (see in [A]).

Fact 1. *Suppose the curve $h = 0$ has only one irreducible branch at infinity and u is a Newton-Puiseux expansion at infinity of this branch. Then*

$$h(x, y) = \prod_{i=1}^{\deg h} (y - u(\epsilon^i x^{\frac{1}{\deg h}}))$$

and $\operatorname{mult}(u) = \deg h$*, where ϵ is a primitive* $\deg h$*-th root of* 1.

Let $\varphi(x, \xi)$ be a finite fractional power series of the form

$$\varphi(x, \xi) = \sum_{k=0}^{n_\varphi - 1} c_k x^{1 - \frac{k}{m_\varphi}} + \xi x^{1 - \frac{n_\varphi}{m_\varphi}}, \tag{1}$$

where ξ is a parameter and $\gcd(\{k = 0, \ldots n_\varphi - 1 : c_k \neq 0\} \cup \{n_\varphi\}) = 1$. Let us represent

$$h(x, \varphi(x, \xi)) = x^{\frac{a_\varphi}{m_\varphi}}(h_0(\xi) + \text{lower terms in } x^{\frac{1}{m_\varphi}}), \ h_0(\xi) \neq 0. \tag{2}$$

The second fact is deduced from the Implicit Function Theorem.

Fact 2. *Let φ and h_0 be as in (1) and (2). If c is a simple zero of $h_0(\xi)$, then there is a Newton-Puiseux expansion at infinity*

$$u(x^{\frac{1}{m_\varphi}}) = \varphi(x, c + \text{lower terms in } x^{\frac{1}{m_\varphi}})$$

for which $h(x, u(x^{\frac{1}{m_\varphi}})) \equiv 0$. Furthermore, $\operatorname{mult}(u)$ *divides m_φ and* $\operatorname{mult}(u) = m_\varphi$ *if $c \neq 0$.*

3. Proof of the Division Lemma. Given $F = (P, Q) \in GA(\mathbf{C}^2)$. We may assume that $\deg P > \deg Q$ and we will prove that $\deg Q$ divides $\deg P$. By choosing a suitable linear coordinate, we can express

$$P(x, y) = y^{\deg P} + \text{lower terms in } y$$

$$Q(x, y) = y^{\deg Q} + \text{lower terms in } y.$$

Observe that F is a polynomial diffeomorphism of $\mathbf{C}^2$ and

$$J(P, Q) := P_x Q_y - P_y Q_x \equiv const. \neq 0.$$

Then, P and Q are reducible and each of the curves $P = 0$ and $Q = 0$ is diffeomorphic to $\mathbf{C}$ which has only one irreducible branch at infinity. Let α and β be the unique irreducible branches at infinity of $P = 0$ and $Q = 0$, respectively. Then, by Fact 1 we can find Newton-Puiseux expansion $u(x^{\frac{1}{\deg P}})$ and $v(x^{\frac{1}{\deg Q}})$ with $\operatorname{mult}(u) = \deg P$ and $\operatorname{mult}(v) = \deg Q$ such that

$$P(x, y) = \prod_{i=1}^{\deg P} (y - u(\sigma^i x^{\frac{1}{\deg P}}))$$

$$Q(x, y) = \prod_{j=1}^{\deg Q} (y - v(\delta^j x^{\frac{1}{\deg P}})),$$

where σ and δ are primitive $\deg P$-th and $\deg Q$-th roots of 1, respectively.

Put $\theta := \min_{ij} \operatorname{ord}(u(\sigma^i x^{\frac{1}{\deg P}}) - v(\delta^j x^{\frac{1}{\deg Q}}))$. Without loss of generality, we can assume $\operatorname{ord}(u(x^{\frac{1}{\deg P}}) - v(x^{\frac{1}{\deg Q}})) = \theta$. We define a fractional power series $\varphi(x, \xi)$ with parameter ξ by deleting in u all terms of order no large than θ and adding to it the term ξx^θ,

$$\varphi(x, \xi) = \sum_{k=0}^{n_\varphi - 1} c_k x^{1 - \frac{k}{m_\varphi}} + \xi x^{1 - \frac{n_\varphi}{m_\varphi}}$$

with $\gcd\{k=0,\dots K-1 : c_k \neq 0\} \cup \{n_\varphi\} = 1$, where $1 - \frac{n_\varphi}{m_\varphi} = \theta$. Then, by definition

$$u(x^{\frac{1}{\deg P}}) = \varphi(x, \xi_u(x)) \text{ with } \xi_u(x) = \alpha_u + \text{ lower terms in } x,$$

$$v(x^{\frac{1}{\deg Q}}) = \varphi(x, \xi_v(x)) \text{ with } \xi_v(x) = \beta_v + \text{ lower terms in } x$$

and $\alpha_u - \beta_v \neq 0$. Let us represent

$$P(x, \varphi(x,\xi)) = x^{\frac{a_\varphi}{m_\varphi}}(P_\varphi(\xi) + \text{ lower terms in } x^{\frac{1}{m_\varphi}})$$

$$Q(x, \varphi(x,\xi)) = x^{\frac{b_\varphi}{m_\varphi}}(Q_\varphi(\xi) + \text{ lower terms in } x^{\frac{1}{m_\varphi}})$$

where a_φ and b_φ are integers and $0 \neq P_\varphi, Q_\varphi \in \mathbf{C}[\xi]$.

Claim 1.
(a) $P_\varphi(\alpha_u) = 0$ *and* $Q_\varphi(\beta_v) = 0$.
(b) *The polynomials* $P_\varphi(\xi)$ *and* $Q_\varphi(\xi)$ *have no common zero.*

Proof. (a) is implied from the equalities $P(x, \varphi(x, \xi_u(x))) = 0$ and $Q(x, \varphi(x, \xi_v(x))) = 0$. For (b), if $P_\varphi(\xi)$ and $Q_\varphi(\xi)$ have a common zero c, then by Fact 2 there exists series

$$\bar{\xi}_u(x) = c + \text{ lower terms in } x,$$

$$\bar{\xi}_v(x) = c + \text{ lower terms in } x$$

such that $\varphi(x, \bar{\xi}_u(x))$ and $\varphi(x, \bar{\xi}_u(x))$ are Newton-Puiseux expansions at infinity of α and β, respectively. For these expansions $\mathrm{ord}(\varphi(x, \bar{\xi}_u(x)) - \varphi(x, \bar{\xi}_v(x)) < \theta$. This contradicts to the definition of u and v. ■

Claim 2. P_φ *and* Q_φ *have only simple zeros.*

Proof. First, observe that

$$a_\varphi > 0, \ b_\varphi > 0. \tag{3}$$

Indeed, for instance, if $a_\varphi \leq 0$, then $F(t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi_v(t^{-m_\varphi}))$ tends to a point $(a, 0) \in \mathbf{C}^2$ as $t \mapsto 0$. This is impossible since F is a diffeomorphism.

Now, let

$$J_\varphi := a_\varphi P_\varphi \frac{d}{d\xi} Q_\varphi - b_\varphi Q_\varphi \frac{d}{d\xi} P_\varphi.$$

Taking differentiation of $DF(t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi)$, by (3) one can get that

$$m_\varphi J(P,Q) t^{n_\varphi - 2m_\varphi - 1} = -J_\varphi t^{-a_\varphi - b_\varphi - 1} + \text{ higher terms in } t.$$

Since $J(P,Q) \equiv const. \neq 0$,

$$J_\varphi \equiv \begin{cases} -m_\varphi J(P,Q), & \text{if } a_\varphi + b_\varphi + n_\varphi = 2m_\varphi \\ 0, & \text{if } a_\varphi + b_\varphi + n_\varphi > 2m_\varphi. \end{cases}$$

If $J_\varphi \equiv 0$, it must be that $P_\varphi^{-b_\varphi} = CQ_\varphi^{-a_\varphi}$ for $C \in \mathbf{C}^*$. This is impossible by Claim 1(b). Thus, $J_\varphi = -m_\varphi J(P,Q)$. In particular, P_φ and Q_φ have only simple zeros. ■

Now, we can complete the proof of the lemma. By Claim 2 the numbers α_u and β_v are simple zero of P_φ and Q_φ, respectively. Then, by Fact 2 there exists Newton-Puiseux expansions at infinity

$$\bar{u}(x^{\frac{1}{m_\varphi}}) = \varphi(x, \alpha_u + \text{ lower terms in } x^{\frac{1}{m_\varphi}}),$$

$$\bar{v}(x^{\frac{1}{m_\varphi}}) = \varphi(x, \beta_v + \text{ lower terms in } x^{\frac{1}{m_\varphi}}),$$

for which $P(x, \bar{u}(x^{\frac{1}{m_\varphi}})) \equiv 0$, $Q(x, \bar{v}(x^{\frac{1}{m_\varphi}})) \equiv 0$ and $\text{mult}(\bar{u})$ and $\text{mult}(\bar{v})$ divide m_φ. Since $\text{mult}(\bar{u}) = \deg P > \deg Q = \text{mult}(\bar{v})$ and $\alpha_u \neq \beta_v$, we get $\alpha_u \neq 0$, $\beta_v = 0$ and $\deg P = m_\varphi$. Hence, $\deg Q | \deg P$. ■

4. Proof of Automorphism Theorem. The proof uses Division Lemma and the following fact which is only an easy elementary excise on homogeneous polynomial.

(*) *Let $f, g \in \mathbf{C}[x,y]$ be homogeneous. If $f_x g_y - f_y g_x \equiv 0$, then there is a homogeneous polynomial $h \in \mathbf{C}[x,y]$ with $\deg h = \gcd(\deg f, \deg g)$ such that*

$$f = ah^{\frac{\deg f}{\deg h}} \text{ and } g = ah^{\frac{\deg g}{\deg h}},\ a, b \in \mathbf{C}^*.$$

(See, for example [E, Lemma 10.2.4, p 253]).

Given $F = (P,Q) \in GA(\mathbf{C}^2)$. Assume, for instance, $\deg P \geq \deg Q$ and $\deg P > 1$. Then, by the Division Lemma $\deg P = m \deg Q$, and hence, by (*) $\deg(P - cQ^m) < \deg P$ for a suitable number $c \in \mathbf{C}$. By induction one can find a finite sequence of automorphisms $\phi_i(x,y)$, $i = 1, \ldots, k$ of the form $(x,y) \mapsto (x + cy^l, y)$ and $(x,y) \mapsto (x, y + cx^n)$ such that the components of the map of $\phi_k \circ \phi_{k-1} \circ \ldots \circ \phi_1 \circ F$ are of degree 1. Note that ϕ_i^{-1} has the form as those of ϕ_i. Then, we get the automorphism Theorem. ■

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References

[A] S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry*, Tata Institute of Fundamental Research, 1977.

[AM] S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. 276 (1975), 148-166.

[BCW] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (1982), 287-330.

[BK] E. Brieskorn, H. Knorrer, *Ebene algebraische Kurven*, Birkhauser, Basel-Boston-Stuttgart 1981.

[CK] J.Chadzy'nski and KT. Krasi'nski, *On a formula for the geometric degree and Jung' theorem* , Universitatis Iagellonicae Acta Mathematica, Fasciculus XXVIII, 1991, 81-84.

[D] W. Dicks, *Automorphisms of the polynomial ring in two variables*, Publ. Sec. Math. Univ Autonoma Barcelona 27, (1983), 155-162.

[E] van den Essen, Arno, *Polynomial automorphisms and the Jacobian conjecture.* (English. English summary) Progress in Mathematics, 190. Birkhauser Verlag, Basel, 2000.

[G] A. Gutwirth, *An inequality for certain pencils of plane curve*, Proc. Amer. Math. Soc. 12 (1961), 631-638.

[J] H. W. E. Jung, *Uber ganze birationale Transformationen der Ebene* , J. Reine Angew. Math. 184 (1942), 161-174.

[K] O. Keller, *Ganze Cremona-Transformationen*, Monatsh. Mat. Phys. 47 (1939), 299-306.

[Ku] W. Van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. (3) 1 (1953), 33-41.

[MW] J. H. McKay and S. S. Wang, *An elementary proof of the automorphism theorem for the polynomial ring in two variables* , J. Pure Appl. Algebra 52 (1988), 91-102.

[N] M. Nagata, *On Automorphism group of* $k[x, y]$, Lectures in Mathematics 5, Tokyo (1972).

[R] R. Rentschler, *Operations du groupe additif sur le plan affine*, C.R.Acad. Sci. Paris Ser. A (1968), 384-387.

[Sh] I.R. Shafarevich, *On some infinite dimensional groups*, Rend.Math. Appl. 25 (1966), 208-212.

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